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NOTE ON n-$(1, r)$-IDEALS OF COMMUTATIVE RINGS

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ABSTRACT. In this paper, we introduce and study n-$(1, r)$-ideals of commutative rings with nonzero identity. Let R be a ring and n be a positive integer. A proper ideal I of R is called an n-$(1, r)$-ideal if whenever nonunit elements $x_1, \ldots, x_n \in R$ and $x_1 \cdots x_{n+1} \in I$, then $x_1 x_2 \cdots x_n \in I$ or $x_{n+1} \in Z(R)$. Various examples and characterizations of n-$(1, r)$-ideals are given. For example, we prove that if R admits an n-$(1, r)$-ideal that is not an $(n-1)$-$(1, r)$-ideal, then R is a local ring. We provide an example of an n-$(1, r)$-ideal that is not an $(n-1)$-$(1, r)$-ideal. In addition, we give a description of n-$(1, r)$-ideals in chained rings. Finally, we study the transfer of n-$(1, r)$-ideals in the localization of rings, the power series rings and the trivial ring extension.

1. Introduction

In this paper, we assume that all rings are commutative with nonzero identity and n a positive integer. If R is a ring and E is an R-module, then the set of zero-divisors of R on E is $Z_R(E) = \{r \in R \mid re = 0 \text{ for some } 0 \neq e \in E\}$. If no confusion can arise, we may delete the R and write $Z(E)$. Also, $Z(R)$ denotes the set of all zero-divisors of R; $\mathrm{Ann}_R(F) := \mathrm{Ann}(F)$, the annihilator of a subset F of R; $\mathrm{Reg}(R) := R \setminus Z(R)$, the set of regular elements of R; $\sqrt{I}$ denotes the radical of an ideal I of R, in the sense of [12, page 17]. For a proper ideal I of R and $x \in R$, the residual of I by x, denoted by $(I :_R x)$ is defined as $\{r \in R \mid rx \in I\}$. A ring R is called a chained ring if either $x \in yR$ or $y \in xR$ for all $0 \neq x, y \in R$.

The prime ideal, a crucial subject in ideal theory, has been thoroughly examined by various authors. In [1], Anderson and Badawi introduced and explored the concept of n-absorbing ideals, representing a generalization of prime ideals. A proper ideal I of R is called an n-absorbing ideal if whenever $x_1 \cdots x_{n+1} \in I$ for some elements $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. In a recent work, Ulucak et al. [15] presented an additional generalization of prime ideals known as n-1-absorbing prime ideals. A proper ideal I of R is said to be an n-1-absorbing prime ideal for some positive integer n if whenever nonunit elements $x_1, \ldots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, then $x_1 \cdots x_n \in I$ or $x_{n+1} \in I$. On the other hand, Mohamadian introduced and investigated the concept of r-ideals in [13].

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Recall that a proper ideal I of R is said to be an r-ideal (resp., pr-ideal) of R if, whenever $ab \in I$ for some $a \in R \setminus Z(R)$ and $b \in R$, then $b \in I$ (resp., $b^n \in I$, for some positive integer n). It is well known that a proper ideal I of R is a pr-ideal if and only if $\sqrt{I}$ is an r-ideal [13, Proposition 2.16]. Recently, Anebri et al. defined a new class of ideals that is related to the class of r-ideals. A proper ideal I of R is said to be a $(1,r)$-ideal if $abc \in I$ for some nonunit elements $a, b, c \in R$, then $ab \in I$ or $c \in Z(R)$.

Let R be a ring and E an R-module. Then $R \propto E$, the *trivial ring extension of R by E*, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in R$ and all $e, f \in E$. (This construction is also known by *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [9], [8]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [2, 4, 7, 10, 11, 14]).

In this paper, our objective is to introduce and explore a new concept of ideals that representing a generalization of prime ideals which consist entirely of zero-divisors of R. A proper ideal I of R is said to be an n-$(1,r)$-ideal if whenever nonunit elements $x_1, \ldots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, then $x_1x_2 \cdots x_n \in I$ or $x_{n+1} \in Z(R)$. In this paper, we present several results to disclose the relationships between this new class and others that already exist. Therefore, our new concept generalizes both r-ideals and n-1-absorbing prime ideals consisting entirely of zero divisors. Additionally, we show in Proposition 2.2 that any n-$(1,r)$-ideal of R is (m,r)-ideal for positive integers m, n with $m \geq n$. Examples 2.3 and 2.4 show that the converses of (1) and (4) in Proposition 2.2 may not be true, respectively. In Theorem 2.8, we show that if a ring R contains an n-$(1,r)$-ideal of R that is not an r-ideal for some positive integer n, then R is a local ring. Moreover, we show in Proposition 2.14 that R is a total quotient ring if and only if every proper ideal of R is n-$(1,r)$-ideal. Also, Theorem 2.15 provides a description of n-$(1,r)$-ideals in chained rings. We prove that if R is a chained ring with maximal ideal M, then I is an n-$(1,r)$-ideal of R if and only if $I = M^n$, $M^{n-1} \subseteq I$ or I is an $(n-1)$-$(1,r)$-ideal of R. Finally, we explore the transfer of n-$(1,r)$-ideals in the localization of rings, the power series rings and the trivial ring extension (see Propositions 2.23, 2.24 and 2.25).

2. Properties of n-$(1,r)$-ideals

We shall begin with the following definition.

Definition 2.1. *Let R be a ring and n be a positive integer. A proper ideal I of R is said to be an n-$(1,r)$-ideal if whenever nonunit elements $x_1, \ldots, x_n \in R$ and $x_1 \cdots x_{n+1} \in I$, then $x_1x_2 \cdots x_n \in I$ or $x_{n+1} \in Z(R)$.*

By definition, it can be seen that a proper ideal I is r-ideal if and only if I is 1-$(1,r)$-ideal of R and $I \subseteq Z(R)$. Also, I is a $(1,r)$-ideal of R if and only if I is a 2-$(1,r)$-ideal of R. In the following proposition, we start by giving some elementary results of n-$(1,r)$-ideals.

Proposition 2.2. *Let R be a ring, I be a proper ideal of R and n be a positive integer. Then the following statements hold:*

(1) *Let $m \geq n$ be positive integers. If I is an n-$(1,r)$-ideal of R, then I is an m-$(1,r)$-ideal of R.*
(2) *The intersection of any family of n-$(1,r)$-ideals of R is an n-$(1,r)$-ideal of R.*
(3) *Every n-1-absorbing prime ideal of R which consists entirely of zero-divisors of R is an n-$(1,r)$-ideal.*
(4) *If I is an n-$(1,r)$-ideal of R, then $\sqrt{I}$ is an $(n-1)$-$(1,r)$-ideal of R. In this case, $x^n \in I$ for all $x \in \sqrt{I} \cap Reg(R)$.*

Proof. (1) We use mathematical induction on n, m. To prove the claim, it is sufficient to prove that I is an $(n+1)$-$(1,r)$-ideal provided I is an n-$(1,r)$-ideal of R. Assume that $x_1 \cdots x_{n+2} \in I$ for some nonunit elements $x_1, \ldots, x_{n+2} \in R$. So, $x_1 \cdots x_{n+1} = (x_1 x_2) x_3 \cdots x_{n+1} \in I$ or $x_{n+2} \in Z(R)$.
(2) and (3) are clair.
(4) Suppose that I is an n-$(1,r)$-ideal of R. Let $x_1, \ldots, x_n$ be nonunit elements of R satisfying $x_1 x_2 \cdots x_n \in \sqrt{I}$, so there exists an integer $k > 0$ such that $x_1^k x_2^k \cdots x_n^k \in I$. This yields that $x_1^{2k} x_2^k \cdots x_n^k \in I$. By hypothesis, we then have $x_1^{2k} x_2^k \cdots x_{n-1}^k \in I$ or $x_n^k \in Z(R)$. Hence $x_1 \cdots x_{n-1} \in \sqrt{I}$ or $x_n \in Z(R)$. Now, let $x \in \sqrt{I} \cap Reg(R)$. Then there exists $k > 0$ such that $x^k \in I$. If $k \leq n$, then we are done. If $k > n$, then $x^k = x \cdots x x^{k-n}$. Hence, by assumption, $x^n \in I$ because $x^{k-n} \notin Z(R)$. This completes the proof. □

We give the following examples to show that the converses of Proposition 2.2 (1) and Proposition 2.2 (4) may not be true.

Example 2.3. *Let (R, M) be a local ring such that $M \neq Z(R)$ and $M^{n+1} \neq M^n$, where n is a positive integer. Then M^{n+1} is an $(n+1)$-$(1,r)$-ideal of R that is not an n-$(1,r)$-ideal. In fact, if $x_1 \cdots x_{n+2} \in M^{n+1}$ for some nonunit elements $x_1, \ldots, x_{n+2} \in R$, then $x_1 \cdots x_{n+1} \in M^{n+1}$. On the other hand, the fact that $M^{n+1} \neq M^n$ and $M \neq Z(R)$ implies that there exist $x_1, \ldots, x_n \in M$ and a regular element $y \in M$ such that $x_1 \cdots x_n \notin M^{n+1}$. Since $x_1 \cdots x_n y \in M^{n+1}$, $x_1 \cdots x_n \notin M^{n+1}$ and $y \notin Z(R)$, we have M^{n+1} is not an n-$(1,r)$-ideal.*

Example 2.4. *Consider the formal power series ring $R = k[[X]]$, where k is a field and X is an indeterminate over k. Then R is a local ring with unique maximal ideal $M = (X)$. By Example 2.3, we know that $(X)^{n+1}$ is not an n-$(1,r)$-ideal. Also, we observe that $\sqrt{(X)^{n+1}} = (X)$ is an $(n-1)$-$(1,r)$-ideal of R.*

Theorem 2.5. *Let R be a ring and I be an n-$(1,r)$-ideal of R for some positive integer n. Then one of the following conditions holds:*

(1) *R is local with maximal ideal $M = \sqrt{I}$ and $M^n \subseteq I$, or*
(2) *I is a pr-ideal of R.*

In addition, if the condition (1) *holds then I is an n-$(1,r)$-ideal.*

Proof. Assume that I is an n-$(1,r)$-ideal of R. Two cases are possible:
Case 1: If $I \nsubseteq Z(R)$, so there exists a regular element $x \in I$. Then, for each nonunit elements $x_1, \dots, x_n \in R$, we have $x_1 \cdots x_n x \in I$ and hence $x_1 \cdots x_n \in I$. It follows that $M^n \subseteq I$ for any maximal ideal M of R and thus $M = \sqrt{M^n} \subseteq \sqrt{I}$. We conclude that $M = \sqrt{I}$ for every maximal ideal M of R. This yields that R is a local ring with maximal ideal $\sqrt{I}$.
Case 2: We suppose that $I \subseteq Z(R)$. Consider two elements $x, y \in R$ such that $xy \in \sqrt{I}$. If x is a unit element of R, then $y \in \sqrt{I}$. If y is a unit element of R, so $x \in \sqrt{I} \subseteq Z(R)$. Now, we assume that x, y are nonunit elements of R and $x \notin Z(R)$, then $x^k y^k \in I$ for some positive integer k. If $k \leq n$, we have $y^n x^k = y \cdots y x^k \in I$. As I is an n-$(1,r)$-ideal, then $y^n \in I$ and so $y \in \sqrt{I}$. If $k > n$, so $y^{k-n+1} y \cdots y x^k \in I$. Since I is an n-$(1,r)$-ideal, we obtain that $y \in \sqrt{I}$. In both cases, we have $\sqrt{I}$ is a *pr*-ideal of R. □

Corollary 2.6. *Let R be a ring and P be a prime ideal of R. Then P is an n-$(1,r)$-ideal for some positive integer n if and only if one of the following conditions holds:*

(1) *$P \subseteq Z(R)$.*
(2) *R is a local ring with maximal ideal P.*

Proof. It suffices to show the "if" assertion. If P is an n-$(1,r)$-ideal of R and $P \nsubseteq Z(R)$, then R is a local ring with maximal ideal M and $M^n \subseteq P$ by Theorem 2.5. Let $x \in M$, so $x^{n+1} \in M^n \subseteq P$. Hence $x \in P$ because P is a prime ideal of R. It follows that $M = P$, as required. □

The following example illustrates that the converse in Theorem 2.5(2) is not true in general.

Example 2.7. *Let R be a local domain with maximal ideal M and $P \subsetneq M$ be a nonzero prime ideal of R. Set $J := 0 \propto P$. It is clear that $J \subseteq Z(R \propto R)$ and $\sqrt{J} = 0 \propto R$ is a prime ideal. Then J is a pr-ideal of $R \propto R$. On the other hand, J is not an n-$(1,r)$-ideal for any positive integer $n > 0$. In fact, let $x \in M \setminus P$ and $0 \neq y \in P$. So, we have $(0,x)(x,0)\cdots(x,0)(y,0) = (0, yx^n) \in J$. However, $(0, x^n) \notin 0 \propto P$ and $(y,0) \notin Z(R \propto R)$.*

Theorem 2.8. *Let R be a non-local ring and n be a positive integer. Then every n-$(1,r)$-ideal of R is an r-ideal.*

Proof. It suffices to prove that every n-$(1,r)$-ideal of R is an $(n-1)$-$(1,r)$-ideal. Suppose that I is an n-$(1,r)$-ideal of R that is not an $(n-1)$-$(1,r)$-ideal. By Theorem 2.5, we may assume that $I \subseteq Z(R)$. Hence there exist nonunit elements $x_1, \dots, x_n \in R$ such that $x_1 \cdots x_n \in I$, $x_1 \cdots x_{n-1} \notin I$ and

$x_n \notin Z(R)$. Let v be a nonunit element of R and $u \in U(R)$. Suppose that $u+v$ is a nonunit element of R. On the one hand, as $vx_1 \cdots x_n \in I$, I is an n-$(1,r)$-ideal of R and $x_n \notin Z(R)$, we conclude that $vx_1 \cdots x_{n-1} \in I$. On the other hand, the fact that $(u+v)x_1 \cdots x_n \in I$ and $x_n \notin Z(R)$ implies that $(u+v)x_1 \cdots x_{n-1} = ux_1 \cdots x_{n-1} + vx_1 \cdots x_{n-1} \in I$ because I is an n-$(1,r)$-ideal of R. It follows that $ux_1 \cdots x_{n-1} \in I$, we conclude that $x_1 \cdots x_{n-1} \in I$, a contradiction. Hence, $u+v$ is a unit element of R. By [5, Lemma 1], we obtain that R is a local ring, which is a contradiction. Thus every n-$(1,r)$-ideal of R is an $(n-1)$-$(1,r)$-ideal. The rest is clear by [3, Theorem 2.7]. □

As immediate consequences of Theorem 2.8, we characterize n-$(1,r)$-ideals in decomposable rings and polynomial rings.

Corollary 2.9. *Let n be a positive integer and I_1 and I_2 be two ideals of the rings R_1 and R_2, respectively. Then the following conditions are equivalent:*

(1) *$I_1 \times I_2$ is an n-$(1,r)$-ideal of $R_1 \times R_2$.*
(2) *$I_1 \times I_2$ is an r-ideal of $R_1 \times R_2$.*

Corollary 2.10. *Let R be a ring, n be a positive integer and I be a proper ideal of R. Then the following assertions are equivalent:*

(1) *$I[X]$ is an n-$(1,r)$-ideal of $R[X]$.*
(2) *$I[X]$ is an r-ideal of $R[X]$.*

Proposition 2.11. *Let I be an n-$(1,r)$-ideal of a ring R for some integer $n > 0$. Assume that I is not an $(n-1)$-$(1,r)$-ideal. Then there exist $(n-1)$ irreducible elements $x_1, \ldots, x_{n-1} \in R$ and a nonunit element $x_n \in R$ such that $x_1 \cdots x_{n-1}x_n \in I$, but neither $x_1 \cdots x_{n-1} \in I$ nor $x_n \in Z(R)$.*

Proof. If I is not an $(n-1)$-$(1,r)$-ideal of R, then there are nonunit elements $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \in I$ but neither $x_1 \cdots x_{n-1} \in I$ nor $x_n \in Z(R)$. Assume that x_i is not an irreducible element for some $i \in \{1, \ldots, n-1\}$. Hence, $x_i = ab$ for some nonunit elements $a, b \in R$. As I is an n-$(1,r)$-ideal of R and $x_1 \cdots x_n = x_1 \cdots x_{i-1}abx_{i+1} \cdots x_n \in I$, we obtain that $x_1 \cdots x_{n-1} \in I$ or $x_n \in Z(R)$, a contradiction. This completes the proof. □

Proposition 2.12. *Let R be a local ring with principal maximal ideal M. If $M = (a_1 \cdots a_{n-1})$ for some positive integer n, then every n-$(1,r)$-ideal contained in $Z(R)$ is an $(n-1)$-$(1,r)$-ideal.*

Proof. Let $x_1, \ldots, x_n$ be nonunit elements of R such that $x_1 \cdots x_n \in I$ and $x_n \notin Z(R)$. By Theorem 2.5, we have I is a pr-ideal of R, which implies that $x_1 \cdots x_{n-1} \in \sqrt{I}$. On the other hand, we have $x_1 \cdots x_{n-1} = ra_1 \cdots a_{n-1}$ for some element $r \in R$. If r is unit, we then have $M = \sqrt{I}$ and so I is an $(n-1)$-$(1,r)$-ideal. If r is a nonunit element, so $ra_1 \cdots a_{n-1}x_n = x_1 \cdots x_n \in I$. Since I is an n-$(1,r)$-ideal of R, we conclude that $ra_1 \cdots a_{n-1} \in I$ and hence $x_1 \cdots x_{n-1} \in I$. Thus I is an $(n-1)$-$(1,r)$-ideal, as required. □

Proposition 2.13. *Let R be a ring and I be a proper ideal of R and n be a positive integer. Then I is an n-$(1, r)$-ideal if and only if whenever $I_1 I_2 \cdots I_{n+1} \subseteq I$ for some proper ideals $I_1, \ldots, I_{n+1}$ of R, then $I_1 \cdots I_n \subseteq I$ or $I_{n+1} \subseteq Z(R)$.*

Proof. Suppose that I is an n-$(1, r)$-ideal and let $I_1, \ldots, I_{n+1}$ be proper ideals of R such that $I_1 I_2 \cdots I_{n+1} \subseteq I$ and $I_{n+1} \nsubseteq Z(R)$. For each $j \in \{1, \ldots, n\}$, let $x_j \in I_j$ and $x_{n+1} \in I_{n+1} \cap Reg(R)$. So $x_1 \cdots x_{n+1} \in I$. By hypothesis, we get $x_1 \cdots x_n \in I$. It follows that $I_1 \cdots I_n \subseteq I$. The converse is clear. □

We call a ring R a total quotient ring if every element of R is either a unit or a zero-divisor. In the following proposition, we give a necessary and sufficient condition, in terms of n-$(1, r)$-ideals, for a ring to be a total quotient ring.

Proposition 2.14. *Let R be a ring and n be a positive integer. Then R is a total quotient ring if and only if every proper ideal of R is an n-$(1, r)$-ideal.*

Proof. The "if" assertion is obvious. Conversely, by Theorem 2.8 and [13, Proposition 3.4], we may assume that R is a local ring with maximal ideal M. In addition, by assumption, we have M^{n+1} is an n-$(1, r)$-ideal and so Theorem 2.5 proves that $M^{n+1} \subseteq Z(R)$ (and hence $M \subseteq Z(R)$) or $M^n = M^{n+1}$. Assume that R is not a total quotient ring, so $M^n = M^{n+1}$. If M^n is a principal ideal, we must have $M^n = 0$ by Nakayama's lemma, a contradiction. On the other hand, let $x \in M^n$. The fact that xR is an n-$(1, r)$-ideal of R and M^n is not a principal ideal implies that $xR \subseteq Z(R)$. It follows that $M^n \subseteq Z(R)$ and thus $M \subseteq Z(R)$, the desired contradiction. We conclude that R is a total quotient ring. □

Theorem 2.15. *Let R be a chained ring with maximal ideal M, and I be a proper ideal of R. Then I is an n-$(1, r)$-ideal of R for some integer $n > 0$ if and only if either $I = M^n$, $M^{n-1} \subseteq I$ or I is an $(n-1)$-$(1, r)$-ideal of R.*

Proof. Let I be an n-$(1, r)$-ideal of R for some integer $n > 0$, so by Theorem 2.5, either ($I \subseteq Z(R)$ and I is a *pr*-ideal) or $M^n \subseteq I$. Now, we suppose that $I \subseteq Z(R)$ and $\sqrt{I}$ is an r-ideal. Let $x_1 \cdots x_n \in I$ for some nonunit elements $x_1, \ldots, x_n \in R$ such that $x_n \notin Z(R)$. Since $\sqrt{I}$ is an r-ideal of R, we have $x_1 \cdots x_{n-1} \in \sqrt{I}$ and hence $x_1 \cdots x_{n-1} \in Z(R)$. This yields that $x_{i_0} \in Z(R)$ for some $i_0 = 1, \ldots, n-1$. As R is a chained ring, we obtain that either $x_{i_0} \in x_n R$ or $x_n \in x_{i_0} R$. As $x_{i_0} \in Z(R)$, we obtain easily that $x_n \notin x_{i_0} R$ and thus $x_{i_0} = a x_n$ for some nonunit element $a \in R$. Since $x_1 \ldots x_{i_0 - 1} a x_n x_{i_0 + 1} \cdots x_n \in I$, $x_n \notin Z(R)$ and I is an n-$(1, r)$-ideal of R, we conclude that $x_1 \cdots x_{n-1} \in I$. Consequently, I is an $(n-1)$-$(1, r)$-ideal of R. On the other hand, we assume that $M^n \subseteq I$ and $M^{n-1} \nsubseteq I$. We will prove that $M^n = I$. Let $x \in I$ and pick an element $x_1 \cdots x_{n-1} \in M^{n-1} \setminus I$. So, $x \in x_1 \cdots x_{n-1} R$ because R is a chained ring. It follows that $x = a x_1 \cdots x_{n-1}$ for some nonunit element $a \in R$ and thus $x = a x_1 \cdots x_{n-1} \in M^n$. Finally, we have that $M^n = I$. □

Theorem 2.16. *Let R be a local ring with maximal ideal M and P be a prime ideal of R. Then PM is an n-$(1,r)$-ideal of R if and only if $P \subseteq Z(R)$ or $P = M$.*

Proof. Suppose that PM is an n-$(1,r)$-ideal of R. Then, by Theorem 2.5, either $M^n \subseteq PM \subseteq P$ or $PM \subseteq Z(R)$. It follows that $M = P$ or $P \subseteq Z(R)$. For the converse, let $x_1, \ldots, x_{n+1}$ be nonunit elements of R such that $x_1 \cdots x_{n+1} \in PM$. If $P = M$, then $x_1 \cdots x_{n-1} \in P$ and so $x_1 \cdots x_n \in PM$. Now, assume that $P \subseteq Z(R)$. If $x_i \in P$ for some $i \in \{1, \ldots, n\}$, then $x_1 \cdots x_n \in PM$. Thus, we may assume that $x_i \notin P$ for any $i \in \{1, \ldots, n\}$. Hence, $x_1 \cdots x_n \notin P$ and so $x_{n+1} \in P$ since $PM \subseteq P$ and P is prime. This yields that $x_{n+1} \in Z(R)$, as required. □

The following lemma is needed in the proof of our next result.

Lemma 2.17. *Let R be a ring, I be an n-$(1,r)$-ideal of R for some integer $n \geq 2$, and $d \in R \setminus (I \cap U(R))$. Then $(I : d) = \{x \in R \mid dx \in I\}$ is an $(n-1)$-$(1,r)$-ideal of R.*

Proof. Assume that I is an n-$(1,r)$-ideal of R with $n \geq 2$, and $x_1 \cdots x_n \in (I : d)$ for some nonunit elements $x_1, \ldots, x_n \in R$, and consequently, $dx_1 \cdots x_n \in I$. By assumption, $dx_1 \cdots x_{n-1} \in I$ or $x_n \in Z(R)$. Thus $x_1 \cdots x_{n-1} \in (I : d)$ or $x_n \in Z(R)$, as needed. □

Let R be a ring and n be a positive integer. An n-$(1,r)$-ideal I of R is said to be a maximal n-$(1,r)$-ideal if there is no n-$(1,r)$-ideal which contains I properly.

Proposition 2.18. *Let R be a ring and n be a positive integer. Then every maximal n-$(1,r)$-ideal of R is an n-1-absorbing prime ideal.*

Proof. Let I be a maximal n-$(1,r)$-ideal of R. Suppose that $x_1 \cdots x_{n+1} \in I$ for some nonunit elements $x_1, \ldots, x_{n+1} \in R$ and $x_{n+1} \notin I$, so by Lemma 2.17, $(I : x_{n+1})$ must be an n-$(1,r)$-ideal. Since I is a maximal n-$(1,r)$-ideal, we obtain that $I = (I : x_{n+1})$ and thus $x_1 \cdots x_n \in I$. □

Proposition 2.19. *Let R be a ring and n be a positive integer. The following statements hold:*

1. *If I is a proper ideal of R and P is an n-1-absorbing prime ideal of R such that $I \cap P$ is an n-$(1,r)$-ideal, then either I or P is an n-$(1,r)$-ideal.*
2. *Suppose that $P_1, ..., P_m$ are n-1-absorbing prime ideals of R, which are not comparable. Then $\cap_{i=1}^m P_i$ is an n-$(1,r)$-ideal if and only if P_i is an n-$(1,r)$-ideal, for all $i = 1, ..., m$.*

Proof. (1) If $I \subseteq P$, then $I = I \cap P$ is an n-$(1,r)$-ideal. Now, we may assume that $I \nsubseteq P$. Take nonunit elements $x_1, \ldots, x_{n+1}$ of R such that $x_1 \cdots x_{n+1} \in P$ and $x_{n+1} \notin Z(R)$. By assumption, there is an element $a \in I \setminus P$, which implies that $(ax_1)x_2 \cdots x_{n+1} \in I \cap P$. So, $x_1 \cdots x_n a \in P$.

As P is an n-1-absorbing prime ideal, we must have $x_1 \cdots x_n \in P$. We conclude that P is an n-$(1,r)$-ideal.
(2) It suffices to show the "if" assertion. Assume that $x_1 \cdots x_{n+1} \in P_i$ for some nonunit elements $x_1, \ldots, x_{n+1} \in R$ and $x_{n+1} \notin Z(R)$. Let $b \in (\prod_{j \neq i} P_j) \setminus P_i$, then $bx_1 \cdots x_{n+1} \in \cap_{j=1}^m P_j$. Since $\cap_{j=1}^m P_j$ is an n-$(1,r)$-ideal, we conclude that $bx_1 \cdots x_n \in \cap_{j=1}^m P_j$, and thus $x_1 \cdots x_n b \in P_i$. This yields that $x_1 \cdots x_n \in P_i$, and so P_i is an n-$(1,r)$-ideal. □

Proposition 2.20. *Let R be a ring, $n \geq 2$ be an integer and $I_1, I_2, \ldots, I_m$ be proper ideals of R such that I_i and I_j are coprime for each $i \neq j$. Then $\cap_{j=1}^m I_j$ is an n-$(1,r)$-ideal if and only if I_j is an $(n-1)$-$(1,r)$-ideal, for each $j \in \{1, \ldots, m\}$.*

Proof. Suppose that $\cap_{j=1}^m I_j$ is an n-$(1,r)$-ideal. Let $x_1, \ldots, x_n$ be nonunit elements of R such that $x_1 \cdots x_n \in I_k$ and $x_n \notin Z(R)$. Since I_k and I_j are coprime for each $k \neq j$, so I_k and $\cap_{j=1, j\neq k}^n I_j$ are coprime. So, $1 = a + b$ with $a \in I_k$ and $b \in \cap_{j=1, j\neq k}^m I_j$. The fact that $\cap_{j=1}^m I_j$ is an n-$(1,r)$-ideal gives that $bx_1 \ldots x_{n-1} \in \cap_{j=1}^m I_j$ since $bx_1 \cdots x_n \in \cap_{j=1}^m I_j$ and $x_n \notin Z(R)$. It follows that $x_1 \cdots x_{n-1} = ax_1 \cdots x_{n-1} + bx_1 \cdots x_{n-1} \in I_k$, and thus I_k is an $(n-1)$-$(1,r)$-ideal. For the converse, it suffices to combine the assertions (1) and (2) of Proposition 2.2. □

Corollary 2.21. *Let R be a ring, $n \geq 2$ be an integer, I be a proper ideal of R and M be a maximal ideal of R. If $I \nsubseteq M$ and $I \cap M$ is an n-$(1,r)$-ideal, then both ideals I and M are $(n-1)$-$(1,r)$-ideals.*

Proof. It can be seen that $I \nsubseteq M$ implies that I and M are coprime ideals. Then, by Proposition 2.20, we have I and M are $(n-1)$-$(1,r)$-ideals. □

According to [13, Definition 3.16], if $R \subseteq T$ are two rings, then we say that R is essential in T, if $R \cap I \neq (0)$, for any nonzero ideal of T.

Proposition 2.22. *Let $R \subseteq S$ be two rings such that R is essential in S and $n > 0$ be an integer. If I is an n-$(1,r)$-ideal of S, then $I \cap R$ is an n-$(1,r)$-ideal of R.*

Proof. Suppose that $x_1 \cdots x_{n+1} \in I \cap R$ for some nonunit elements $x_1, \ldots, x_{n+1} \in R$ and $x_{n+1} \notin Z(R)$. We will prove that $x_{n+1} \notin Z(S)$. If $x_{n+1} \in Z(S)$, then $Ann_S(c) \neq 0$. So, by hypothesis, $Ann_R(c) = Ann_S(c) \cap R \neq (0)$. This implies that $x_{n+1} \in Z(R)$, which is a contradiction. Now, since I is an n-$(1,r)$-ideal of S and $x_1 \cdots x_{n+1} \in I$, we have that $x_1 \cdots x_n \in I$. It gives that $x_1 \cdots x_n \in I \cap R$. This completes the proof. □

Proposition 2.23. *Let R be a ring, n be a positive integer and S be a multiplicatively closed subset of R. If I is an n-$(1,r)$-ideal of R, then $S^{-1}I$ is an n-$(1,r)$-ideal of $S^{-1}R$.*

Proof. Suppose that I is an n-$(1,r)$-ideal of R. Let $\frac{x_1}{s_1}, \ldots, \frac{x_{n+1}}{s_{n+1}}$ be nonunit elements of $S^{-1}R$ such that $\frac{x_1}{s_1} \cdots \frac{x_{n+1}}{s_{n+1}} = \frac{x_1 \cdots x_{n+1}}{s_1 \cdots s_{n+1}} \in S^{-1}I$. Then there exists an element $t \in S$ such that $tx_1 \cdots x_{n+1} \in I$, which implies that $tx_1 \cdots x_n \in I$ or $x_{n+1} \in Z(R)$ since I is an n-$(1,r)$-ideal. It follows that $\frac{x_1 \cdots x_n}{s_1 \cdots s_n} \in S^{-1}I$ or $\frac{x_{n+1}}{s_{n+1}} \in Z(S^{-1}R)$. □

Proposition 2.24. *Let I be a proper ideal of a ring R. Then $I + XR[[X]]$ is an n-$(1,r)$-ideal for some positive integer n if and only if R is a local ring with maximal ideal $M = \sqrt{I}$ and $M^n \subseteq I$.*

Proof. Assume that $I + XR[[X]]$ is an n-$(1,r)$-ideal for some integer $n > 0$. So, by Theorem 2.5, we have $R[[X]]$ is a local ring with maximal ideal $L = \sqrt{I + XR[[X]]} = \sqrt{I} + XR[[X]]$ and $L^n \subseteq I + XR[[X]]$. Using [6, Theorem 2], we conclude that R is local with maximal ideal $M = \sqrt{I}$. In addition, it can be seen that $M^n \subseteq I$. Conversely, suppose that R is local with maximal ideal $M = \sqrt{I}$ and $M^n \subseteq I$. By [6, Theorem 2], we obtain that $R[[X]]$ is local with maximal ideal $L = M + XR[[X]]$. Moreover, $L^n \subseteq M^n + XR[[X]] \subseteq I + XR[[X]]$. Thus $I + XR[[X]]$ is an n-$(1,r)$-ideal of R. □

Proposition 2.25. *Let R be a ring and E be an R-module. If I is an n-$(1,r)$-ideal of R for some integer $n > 0$, then $I \propto E$ is an n-$(1,r)$-ideal of $R \propto E$. The converse is true if $Z(E) \subseteq Z(R)$.*

Proof. Suppose that I is an n-$(1,r)$-ideal of R for some integer $n > 0$. Let $(x_1, e_1), \ldots, (x_{n+1}, e_{n+1})$ be nonunit elements of $R \propto E$ such that $(x_1, e_1) \cdots (x_{n+1}, e_{n+1}) \in I \propto E$. So, $x_1 \cdots x_{n+1} \in I$ and $x_1, \ldots, x_{n+1}$ are nonunit elements of R. The fact that I is an n-$(1,r)$-ideal of R proves that $x_1 \cdots x_n \in I$ or $x_{n+1} \in Z(R)$ and hence $(x_1, e_1) \cdots (x_n, e_n) \in I \propto E$ or $(x_{n+1}, e_{n+1}) \in Z(R \propto E)$. So, we conclude that $I \propto E$ is an n-$(1,r)$-ideal of $R \propto E$. Now, we will prove the converse under the additional condition $Z(E) \subseteq Z(R)$. If I is a proper ideal of R such that $I \propto E$ is an n-$(1,r)$-ideal of R. Suppose that $x_1 \cdots x_{n+1} \in I$ for some nonunit elements of R, then $(x_1, 0) \cdots (x_{n+1}, 0) \in I \propto E$ and $(x_1, 0), \ldots, (x_{n+1}, 0)$ are nonunit elements of $R \propto E$. Thus, $(x_1, 0) \cdots (x_n, 0) \in I \propto E$ or $(x_{n+1}, 0) \in Z(R \propto E)$. By hypothesis, we have $x_1 \cdots x_n \in I$ or $x_{n+1} \in Z(R)$ and hence I is an n-$(1,r)$-ideal of R. This completes the proof. □

In Proposition 2.25, the converse may not be true if one deletes the hypothesis that $Z(E) \subseteq Z(A)$.

Example 2.26. *Let p be a positive prime integer and consider the $\mathbb{Z}$-module $E = \mathbb{Z}_p$. Then $p\mathbb{Z} \propto E$ is an n-$(1,r)$-ideal of $\mathbb{Z} \propto E$ because it is a prime ideal and $p\mathbb{Z} \propto E \subseteq \mathbb{Z} \propto E$. However, by Theorem 2.5, $p\mathbb{Z}$ is not an n-$(1,r)$-ideal of $\mathbb{Z}$ for each positive integer n.*

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